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ON COMPOSITE STEADY GRAVITATIONAL WAVES OF FINITE AMPLITUDE

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The problem of plane steady waves of finite amplitude generated by pressure periodically distributed over the surface of a heavy fluid of infinite depth was first formulated by Stretenskii in 1953 [1] who also gave its approximate solution. The exact solution of this problem was presented by the Author for an infinitely deep stream in papers [2, 3] and also for streams of finite depth in [4–6].

All of these papers had investigated waves which cease to exist when the periodic part of the pressure distributed over the stream surface vanishes and the flow becomes uniform. We shall call such waves induced. Waves of constant amplitude occurring at particular flow velocities under conditions of constant pressure over the whole surface will be called free waves. An exact solution of the problem of these waves was first given by Nekrasov in 1921 [7].

The possibility of a simultaneous occurrance of these two kinds of waves of small finite amplitude in a steam of infinite depth at particular flow velocities is established below. We shall call such waves composite waves. When the periodic part of the pressure distributed over the surface vanishes, such waves are transformed into free waves.

The general method of computation of characteristics of these waves is presented. The complete computation of the first three approximations, and an approximate equation of the wave profile are given.

1. Statement of problem and derivation of the fundamental equation. We shall consider a plane-parallel stationary motion of a perfect incompressible heavy fluid bounded from above by a free surface under pressure $p = p_0' + p_0(x)$, where $p_0' = \mathrm{const}$, and $p_0(x)$ is a specified periodic function of the horizontal coordinate x. We shall assume that the fluid flows from left to right at a given constant velocity c at an infinite depth. Induced waves, as had been already stated in the introduction, do occur at any velocity c in the presence of the $p_0(x)$ term, while in the absence of $p_0(x)$ free waves appear at special values of c.

We shall assume that the pressure on the free surface contains these two components, and that in these conditions the form of the free surface in a system of coordinates attached to a progressing wave moving from right to left at a certain special velocity, is that of a periodic steady wave. If these waves do not vanish at $p_0(x) \equiv 0$, we shall call them, as previously indicated, composite waves.

Let the looked for composite wave and pressure $p_0(x)$ be equally symmetric about the vertical through the wave crest. We superpose the y-axis on the axis of symmetry and direct it upwards. We locate the coordinate origin O at the intersection point of the y-axis with the free surface, and direct the x-axis towards the right.

The xy-plane of flow will be taken as the plane of the complex variable $z = x + iy$. We introduce the usual notation: φ the velocity potential, ψ the stream function, $w = \varphi + i\psi$ the complex potential of velocities, and U and V the projections of the velocity vector $\mathbf{q}$ onto the coordinate axes. We then have

$$\frac{dw}{dz} = -U + iV, \qquad U = -\frac{\partial\varphi}{\partial x}, \quad V = -\frac{\partial\varphi}{\partial y}$$

For the derivation of the problem fundamental equation from the boundary condition we shall first map the region occupied by one wave represented by an infinite vertical half-strip bounded at the top by a wavelike curve onto the half-strip $0 \leqslant \varphi \leqslant c\lambda$, $0 \leqslant \psi \leqslant \infty$ in the w-plane, and then map the latter onto the interior of a unit circle with its center at the coordinate origin of plane $u = u_1 + iu_2$. The wavelength λ is assumed to coincide with the period of function $p_0(x)$.

The latter conformal mapping is given by formula

$$w = \frac{\lambda c}{2\pi i} \ln u \tag{1.1}$$

As the result of this the wave profile is transformed into the unit circle circumference slit along radius $\arg u = 0$.

The mapping of circle $|u| \leqslant 1$ onto the single wave plane z is carried out by means of formula

$$\frac{dz}{du} = -\frac{\lambda}{2\pi i}\frac{f(u)}{u}, \qquad f(u) = 1 + \sum_{k=1}^{\infty} a_k u^k \tag{1.2}$$

The a_k coefficients are real because of the wave symmetry relative to the y-axis, and $a_0 = 1$ due to the stream velocity at infinity being equal to c and directed along the x-axis.

Considering that at the surface $p = p_0' + p_0(x)$ and using the Bernoulli surface integral, we find from (1.2) by differentiating with respect to θ

$$\frac{1}{\rho}\frac{dp_0}{dx}\frac{dx}{d\theta} = -g\frac{dy}{d\theta} - \frac{1}{2}\frac{dq^2}{d\theta} \qquad (u = e^{i\theta}) \tag{1.3}$$

Here θ is the position vector angle with the u_1-axis, ρ is the density, g is the acceleration of gravity and q is the module of velocity vector $\mathbf{q}$.

Introducing, as usual, function [7]

$$\omega(u) = \Phi + i\tau = -i \ln f(u) \tag{1.4}$$

we obtain by virtue of (1.1) and (1.2)

$$\frac{dw}{dz} = -ce^{-\tau + i\Phi} \tag{1.5}$$

Hence, function Φ is throughout the stream equal to the angle between vector $\mathbf{q}$ and the x-axis, and

$$q = ce^{\tau} \tag{1.6}$$

From (1.4) and (1.2) we find that for $u = e^{i\theta}$

$$\frac{dx}{d\theta} + i\frac{dy}{d\theta} = -\frac{\lambda}{2\pi} e^{-\tau(\theta)} (\cos\Phi + i\sin\Phi) \tag{1.7}$$

By virtue of this and of (1.6) we obtain from Eq. (1.3)

$$\frac{d}{d\theta} e^{3\tau} = \frac{3g\lambda}{2\pi c^2}\left(\sin\Phi + \frac{1}{\rho g}\frac{dp_0}{dx}\cos\Phi\right) \tag{1.8}$$

or after integration

$$e^{3\tau} = \frac{3g\lambda}{2\pi c^2 \mu}\left[1 + \int_0^\theta \left(\sin\Phi + \frac{1}{\rho g}\frac{dp_0}{dx}\cos\Phi\right) d\eta\right] \tag{1.9}$$

where μ^{-1} is the constant of integration, and

$$\mu = \frac{3g\lambda}{2\pi c^2} e^{-3\tau(0)} \tag{1.10}$$

Parameter μ is associated at the surface to p_0' by the additive constant of p.

From (1.9) we have

$$\frac{d\tau}{d\theta} = \frac{\mu H[\Phi(\theta), Q(\theta)]}{3}\left[1 + \mu\int_0^\theta H[\Phi, Q]\, d\eta\right]^{-1} \tag{1.11}$$

where

$$H[\Phi(\theta), Q(\theta)] = \sin\Phi(\theta) + Q(\theta)\cos\Phi(\theta) \tag{1.12}$$

$$Q(\theta) = \frac{1}{\rho g}\frac{dp_0}{dx}$$

Equality (1.11) defines the relationship between functions $\tau(\theta)$ and $\Phi(\theta)$ along the circumference $|u| = 1$.

As function $\tau(\theta)$ is symmetric about the real axis, hence $\tau(\theta) = \tau(2\pi - \theta)$.

From this follows the known Dini's relation

$$\Phi(\theta) = 3\int_0^{2\pi} \frac{d\tau}{d\eta} K(\eta, \theta)\, d\eta \tag{1.13}$$

The arbitrary constant has been omitted here because the tangent to the wave crest is horizontal. The kernel $K(\eta, \theta)$ is of the form

$$K(\eta, \theta) = -\frac{1}{6\pi}\ln\left|\frac{\sin {}^1\!/_2(\eta - \theta)}{\sin {}^1\!/_2(\eta + \theta)}\right| = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\eta \sin n\theta}{3n} \tag{1.14}$$

It follows from (1.14) that the normalized eigenfunctions $\varphi_n(\theta)$ and the eigenvalues ν_n of kernel $K(\eta, \theta)$ are defined by formulas

$$\varphi_n(\theta) = \frac{\sin n\theta}{\sqrt{\pi}}, \qquad \nu_n = 3n \tag{1.15}$$

Finally, from (1.11) and (1.13) we have

$$\Phi(\theta) = \mu \int_0^{2\pi} H[\Phi(\eta),\ Q(\eta)] \left[1 + \mu \int_0^{\eta} H[\Phi,\ Q]\, d\eta_1\right]^{-1} K(\eta, \theta)\, d\eta \qquad (1.16)$$

which is the integral equation of this problem. From it we obtain for $p = p_0' = \text{const}$ the known Nekrasov's equation [7].

When solving the composite wave problem we shall assume that

$$Q(\theta) = \frac{1}{\rho g}\frac{dp_0}{dx} = \sum_{n=1}^{\infty} \varepsilon^{n+2} d_n \sin n\theta \qquad (1.17)$$

Here ε is a small dimensionless parameter and d_n are given real numbers, with expansion

$$\varepsilon^3 d_1 + \varepsilon^4 d_2 + \ldots + \varepsilon^n d_n + \ldots$$

convergent in a circle of radius $\varepsilon_0 > 0$; $d_1 < 0$ is assumed (see Note in Sect. 3).

We recall that in the initial statement of the problem $p_0(x)$ was assumed to be a given periodic function of x. It can be however shown that the solution of the problem here considered with condition (1.17) taken into account is equivalent to specifying expansion

$$\frac{1}{\rho g}\frac{dp_0}{dx} = -\sum_{n=1}^{\infty} \varepsilon^{n+2} c_n' \sin\frac{2\pi n}{\lambda}x \qquad \left(c_n' = \sum_{m=0}^{\infty} \varepsilon^m c_{mn}'\right)$$

Here coefficients c_{0n}' may either be considered as given and used for the computation of d_n, or vice versa; coefficients c_{mn}' $(m = 1, 2, \ldots)$ are determined by d_n (see (3.7) in Sect. 3).

If however one assumed $d_n = d_{0n} + d_{1n}\varepsilon + d_{2n}\varepsilon^2 + \ldots$ (this is not the case here), then c_{mn}' $(m = 1, 2, 3, \ldots)$ may be taken as given and used for determining d_{in} $(i = 1, 2, \ldots)$, or vice versa.

We also note that function $Q(\theta)$ of the form of (1.17) may be specified for the determination of an induced wave in the case of $\mu_0 = \nu_1$; this would yield the solution in the form of a series expansion in integral powers of ε.

We transform Equality (1.10). Function $\tau(\theta)$ will be even along circle $|u| = 1$, and function $\Phi(\theta)$ will be odd. Hence we assume

$$\Phi(\theta) = \sum_{k=1}^{\infty} b_k \sin k\theta, \qquad \tau(\theta) = -\sum_{k=1}^{\infty} b_k \cos k\theta \qquad (1.18)$$

Here $b_0 = 0$ because $\omega(0) = 0$ (see (1.2) and (1.4)).

Consequently (1.10) will be of the form

$$\mu = \mu_0 \exp\left(3 \sum_{k=1}^{\infty} b_k\right), \qquad \mu_0 = \frac{3g\lambda}{2\pi c^2} \qquad (1.19)$$

The equation of the form of (1.19) was used for the determination of μ in the induced wave problem in [2], where the solution is given for the cases of $\mu_0 \neq \nu_n$ and $\mu_0 = \nu_1$. For a free wave Eq. (1.19) is of the form

$$\mu = \mu_0 (1 - \varepsilon^2) \exp\left(3 \sum_{k=1}^{\infty} b_k\right), \qquad \mu_0 = \nu_1 = \frac{3g\lambda}{2\pi c_*^2} \qquad (1.20)$$

if we assume

$$\frac{1}{c^2} = \frac{1}{c_*^2}(1 - \varepsilon^2) \qquad (1.21)$$

where c_* is the initial wave velocity.

We note that Nekrasov [7] had specified $\mu = \nu_1 + \mu'$ and found the solution, including that of $1/c^2$ in the form of expansions in powers of μ'. Specifying $1/c^2$ in the form of (1.21), the solution, including that of μ, has the form of expansions in powers of ε. In order to pass from one form of solution to the other it is necessary to express parameter μ' in terms of ε, or vice versa.

When considering a composite wave, we shall express $1/c^2$ by means of formula (1.21), and consequently, μ by (1.20) with ε being the same small parameter as in (1.17).

It will be seen from (1.21) that ε^2 defines a small variation of the specified value of $1/c_*^2$. We could have substituted in (1.21) factor $(1 - \alpha^2\varepsilon^2)$ for $(1 - \varepsilon^2)$. This would have resulted in the solution yielding an induced wave for $\alpha = 0$ and $Q(\theta) \not\equiv 0$. However in order to simplify computations we have assumed $\alpha = 1$. Assuming further that

$$\Psi(\theta) = \left[1 + \mu\int_0^\theta H[\Phi(\eta),\ Q(\eta)]\,d\eta\right]^{-1} \tag{1.22}$$

where $H[\Phi,\ Q]$ conforms to (1.12), we reduce Eq. (1.16), as in the case of Nekrasov's equation [7], to the following equivalent system of two equations for the unknown functions $\Phi(\theta)$ and $\Psi(\theta)$:

$$\Phi(\theta) = \mu\int_0^{2\pi} K(\eta, \theta)\, H[\Phi(\eta),\ Q(\eta)]\,\Psi(\eta)\,d\eta \tag{1.23}$$

$$\Psi(\theta) = 1 - \mu\int_0^\theta \Psi^2(\eta)\, H[\Phi(\eta), Q(\eta)]\,d\eta \tag{1.24}$$

The problem is thus reduced to the determination of functions $\Phi(\theta, \varepsilon)$ and $\Psi(\theta, \varepsilon)$, and of parameter $\mu(\varepsilon)$ satisfying Eq. (1.23), (1.24) and (1.20).

2. Derivation of solution of the basic system in the form of series expansions in powers of ε. We shall look for the solution of the basic system and of parameter μ in the form of series expansions in powers of ε as follows:

$$\Phi(\theta) = \sum_{n=1}^\infty \varepsilon^n\Phi_n(\theta), \qquad \Psi(\theta) = \sum_{n=0}^\infty \varepsilon^n\Psi_n(\theta),\ \Psi_0 = 1, \qquad \mu = \mu_0 + \sum_{n=1}^\infty \varepsilon^n\mu_n \tag{2.1}$$

In order to derive the equation which would make possible the determination of coefficients of the first two of these expansions it is necessary to substitute expansions (2.1) into (1.23) and (1.24), and then expand the result into series in powers of ε and equate in the obtained relationships coefficients of same ε^n.

This necessitates the expansion of $\sin\Phi(\eta)$ and $\cos\Phi(\eta)$ into series of powers of ε, taking into account (2.1). The first of these expansions will be of the form

$$\sin\Phi(\eta) = \sum_{m=1}^\infty A_{2m-1}\Phi^{2m-1} = \sum_{m=1}^\infty A_{2m-1}\left[\sum_{i=1}^\infty \Phi_i\varepsilon^i\right]^{2m-1} = \sum_{n=1}^\infty s_n\varepsilon^n$$

$$\left(A_{2m-1} = \frac{(-1)^{m+1}}{(2m-1)!}\right) \tag{2.2}$$

If we do not substitute values of A_{2m-1} at $m > 1$, and because of $A_1 = 1$, we have

$$s_1 = \Phi_1,\quad s_2 = \Phi_2,\qquad s_3 = \Phi_3 + A_3\Phi_1^3,\qquad s_4 = \Phi_4 + A_3 3\ \Phi_1^2\Phi_2$$

$$s_5 = \Phi_5 + A_3 3\Phi_1^2\Phi_2^2 + A_3 3\Phi_3\Phi_1^2 + A_5\Phi_1^5$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$s_n = \sum_{m=1}^{m\leqslant 1/2(n+1)} A_{2m-1}\Big(\sum_{\alpha_1,\ldots,\alpha_n} \frac{(2m-1)!}{\alpha_1!\,\alpha_2!\ldots\alpha_n!}\Phi_1^{\alpha_1}\Phi_2^{\alpha_2}\ldots\Phi_n^{\alpha_n}\Big) \tag{2.3}$$

Here α_i are positive integers including zero which satisfy relations

$$\alpha_1+\alpha_2+\ldots+\alpha_n = 2m-1, \qquad \alpha_1+2\alpha_2+\ldots+n\alpha_n = n \tag{2.4}$$

The second expansion is of the form

$$\cos\Phi(\eta) = \sum_{m=0}^{\infty} A_{2m}\Phi^{2m} = \sum_{m=0}^{\infty} A_{2m}\Big[\sum_{i=1}^{\infty}\Phi_i\varepsilon^i\Big]^{2m} = \sum_{n=0}^{\infty} c_n\varepsilon^n \quad \Big(A_{2m} = \frac{(-1)^m}{2m!}\Big) \tag{2.5}$$

If we do not substitute values of A_{2m} at $m > 1$, and because of $A_0 = 1$, we obtain

$$c_0 = 1, \quad c_1 = 0, \; c_2 = A_2\Phi_1^2, \quad c_3 = A_2 2\Phi_1\Phi_2$$

$$c_4 = A_4\Phi_1^4 + A_2 2\Phi_1\Phi_3 + A_2\Phi_2^2$$

$$c_5 = A_4 4\Phi_1^3\Phi_2 + A_2 2\Phi_4\Phi_1 + A_2 2\Phi_2\Phi_3$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$c_n = \sum_{m=1}^{m\leqslant 1/2 n} A_{2m}\Big(\sum_{\alpha_1,\ldots,\alpha_n} \frac{2m!]}{\alpha_1!\alpha_2!\ldots\alpha_n!}\Phi_1^{\alpha_1}\Phi_2^{\alpha_2}\ldots\Phi_n^{\alpha_n}\Big) \tag{2.6}$$

Relations (2.4) hold for α_i, if in the first of these $2m$ is substituted for $2m-1$.

We revert to Eq. (1.20) which will be used for defining μ (ε) in the form of expansion (2.1). We assume $\mu_0 = \nu_1$, and take μ_n ($n = 1,2,3,\ldots$) as defined by (1.20). We shall adduce the recurrent formulas used for the determination of μ_i. It follows from the construction of function Φ and (1.18) that

$$b_1 = \sum_{n=1}^{\infty} b_{1n}\varepsilon^n, \quad b_2 = \sum_{n=2}^{\infty} b_{2n}\varepsilon^n, \ldots, b_k = \sum_{n=k}^{\infty} b_{kn}\varepsilon^n$$

If we assume that $b_{ij} = 0$ for $j < i$, we obtain

$$b_1 = \sum_{n=1}^{\infty} b_{1n}\varepsilon^n, \quad b_2 = \sum_{n=1}^{\infty} b_{2n}\varepsilon^n, \ldots, b_k = \sum_{n=1}^{\infty} b_{kn}\varepsilon^n$$

Hence

$$b_1+b_2+\ldots+b_k+\ldots = \sum_{n=1}^{\infty}(b_{1n}+b_{2n}+\ldots+b_{nn})\,\varepsilon^n \tag{2.7}$$

$$\exp[3(b_1+b_2+\ldots+b_k+\ldots)] = \exp\Big[3\sum_{n=1}^{\infty}(b_{1n}+b_{2n}+\ldots+b_{nn})\,\varepsilon^n\Big]$$

Let

$$\exp\Big[3\sum_{n=1}^{\infty}(b_{1n}+b_{2n}+\ldots+b_{nn})\,\varepsilon^n\Big] = \sum_{n=0}^{\infty}\varepsilon^n e_n \tag{2.8}$$

Then it can be shown that

$$e_n = \frac{3}{n}\sum_{j=1}^{n} j\,(b_{1j}+b_{2j}+\ldots+b_{jj})\,e_{n-j} \qquad (e_0 = 1) \tag{2.9}$$

Substituting into (1.20) the expansion of μ and expansion (2.8) and equating coefficients of ε^n, we find

$$\mu_n = \mu_0\,(e_n - e_{n-2}) \tag{2.10}$$

We have to assume here that $e_{-2} = e_{-1} = 0$.

We pass to the solution of equations of $\Phi_n(\theta)$, $\Psi_n(\theta)$ and μ_n. These equations are derived in the manner indicated above by using the adduced expansions. Omitting intermediate computations, we write these equations in their final form.

2.1. Determination of $\Phi_1(\theta)$, $\Psi_1(\theta)$ and μ_1. For these functions we have the system

$$\Phi_1(\theta) = \mu_0 \int_0^{2\pi} K(\eta, \theta)\Phi_1(\eta)\, d\eta, \qquad \Psi_1(\theta) = -\mu_0 \int_0^{\theta} \Phi_1(\eta)\, d\eta \quad (2.11)$$

As $\mu_0 = \nu_1 = 3$, therefore the solution of the first equation which is a homogeneous linear Fredholm's integral equation of the second kind, is

$$\Phi_1(\theta) = C_{11}\sin\theta, \qquad C_{11} = \frac{C_{11}'}{\sqrt{\pi}} \quad (2.12)$$

Constant C_{11}, as will be shown in the following, is determined by the solvability condition of the equation of $\Phi_3(\theta)$.

Substituting $\Phi_1(\theta)$ defined by (2.12) into the second of Eqs. (2.11) we find

$$\Psi_1(\theta) = -\mu_0 C_{11}\int_0^{\theta} \sin\eta\, d\eta = 3C_{11}(\cos\theta - 1) \quad (2.13)$$

As $e_0 = 1$, hence for the determination of μ_1 we have from (2.10)

$$e_1 = 3b_{11}e_0 = 3C_{11}, \qquad \mu_1 = \mu_0 e_1 = 9C_{11} \quad (2.14)$$

2.2. Determination of $\Phi_2(\theta)$, $\Psi_2(\theta)$ and μ_2. These functions are to be derived from system

$$\Phi_2(\theta) = \mu_0 \int_0^{2\pi} K(\eta, \theta)\,[\Phi_2(\eta) + P_2(\eta)]\, d\eta \quad (2.15)$$

$$\Psi_2 = -\int_0^{\theta} [\mu_0\Phi_2(\eta) + \Phi_1(\eta)(\mu_1 + 2\mu_0\Psi_1(\eta))]\, d\eta \quad (2.16)$$

where

$$P_2(\eta) = {}^3/_2 C_{11}^2 \sin 2\eta \quad (2.17)$$

By adding $P_2(\theta)$ to the two sides of Eq. (2.15) we obtain its equivalent equation. By the third of Fredholm's theorems the solvability condition of the latter, and consequently also of Eq. (2.15) is of the form

$$\int_0^{2\pi} P_2(\eta)\frac{\sin\eta}{\sqrt{\pi}}\, d\eta = 0 \quad (2.18)$$

It is fulfilled by virtue of (2.17). Hence the solution of Eq. (2.15) is

$$\Phi_2(\theta) = C_{12}\sin\theta + \mu_0 \sum_{i=2}^{\infty} \frac{a_{i2}\varphi_i(\theta)}{\nu_i - \mu_0}, \qquad C_{12} = \frac{C_{12}'}{\sqrt{\pi}}$$

$$a_{i2} = \int_0^{2\pi} P_2(\eta)\varphi_i(\eta)\, d\eta \quad (2.19)$$

Constant C_{12} is defined by the solvability condition of $\Phi_4(\theta)$.

By virtue of (2.17) we obtain from (2.19)

$$a_{i2} = 0 \quad (i = 3, 4, 5, \ldots) \qquad a_{22} = {}^3/_2\, C_{11}^2 \sqrt{\pi} \quad (2.20)$$

Substituting these expressions into formula (2.19), we obtain

$$\Phi_2(\theta) = C_{12}\sin\theta + C_{22}\sin 2\theta \qquad (C_{22} = {}^3/_2 C_{11}^2) \tag{2.21}$$

Substituting $\Phi_2(\theta)$ from (2.21) together with obtained values of the remaining magnitudes into Eq. (2.16), we obtain after necessary computations

$$\Psi_2(\theta) = (3C_{12} - 9C_{11}^2)(\cos\theta - 1) + {}^{27}/_4 C_{11}^2(\cos 2\theta - 1) \tag{2.22}$$

In order to determine μ_2 we set $n = 2$ in (2.9) and (2.10), then

$$e_2 = {}^3/_2 [b_{11}e_1 + 2(b_{12} + b_{22})e_0], \qquad \mu_2 = \mu_0(e_2 - e_0) \tag{2.23}$$

Because

$$b_{11} = C_{11}, \qquad b_{12} = C_{12}, \qquad b_{22} = C_{22} \tag{2.24}$$

and taking into account (2.14) we obtain from (2.23)

$$e_2 = 3(3C_{11}^2 + C_{12}), \qquad \mu_2 = 3[3(3C_{11}^2 + C_{12}) - 1] \tag{2.25}$$

2.3. Determination of $\Phi_3(\theta)$, $\Psi_3(\theta)$ and μ_3. In this case we obtain the system

$$\Phi_3(\theta) = \mu_0 \int_0^{2\pi} K(\eta,\theta)[\Phi_3(\eta) + P_3(\eta)]\,d\eta \tag{2.26}$$

$$\Psi_3(\theta) = -\int_0^{\theta} [\mu_0\Phi_3(\eta) + \mu_0 A_3\Phi_1^3 + \Phi_2(\mu_1 + 2\mu_0\Psi_1) + \Phi_1(2\mu_1\Psi_1 + \mu_0\Psi_1^2 + 2\mu_0\Psi_2 + \mu_2) + d_1\mu_0\sin\eta]\,d\eta \tag{2.27}$$

where

$$P_3(\eta) = (C_{11}^3 - C_{11} + d_1)\sin\eta + 3C_{11}C_{12}\sin 2\eta + {}^{17}/_3 C_{11}^3\sin 3\eta \tag{2.28}$$

The solvability condition for Eq. (2.26), similar to that for (2.18), reduces to specifying the absence in Eq. (2.28) of the term containing $\sin\eta$. Equating the coefficients of this term to zero, we have

$$C_{11}^3 - C_{11} + d_1 = 0 \tag{2.29}$$

As will be explained later (see Note at the end of Sect. 3), the nature of this problem necessitates that $C_{11} > 0$. Analysis of the roots of the incomplete cubic equation (2.29) had shown that only one root $C_{11} > 0$ corresponds to $d_1 < 0$. We would point out that for $d_1 = 0$ (and because $C_{11} \neq 0$) Eq. (2.29) reduces to a quadratic equation which defines C_{11} in the case of a free wave.

With condition (2.29) satisfied, the solution of Eq. (2.26) is obtained from a formula analogous to (2.19), and is if the form

$$\Phi_3(\theta) = C_{13}\sin\theta + C_{23}\sin 2\theta + C_{33}\sin 3\theta \tag{2.30}$$

where

$$C_{13} = C_{13}'/\sqrt{\pi}, \qquad C_{23} = 3C_{11}C_{12}, \qquad C_{33} = {}^{17}/_6 C_{11}^3 \tag{2.31}$$

is assumed.

After substitution into (2.27) of all of the derived values and completion of computation, we obtain

$$\Psi_3(\theta) = 3(C_{13} - C_{11} + {}^{35}/_8 C_{11}^3 - 6C_{11}C_{12} + d_1)(\cos\theta - 1) + {}^{27}/_2(C_{11}C_{12} - {}^3/_2 C_{11}^3)(\cos 2\theta - 1) + {}^{131}/_8 C_{11}^3(\cos 3\theta - 1) \tag{2.32}$$

Using (2.9) and (2.10) with $n = 3$, we obtain after computation

$$\mu_3 = {}^{159}/_2 C_{11}^3 + 54C_{11}C_{12} + 9C_{13} - 9C_{11} \tag{2.33}$$

2.4. Determination of further approximations. We do not adduce further computations. We shall however indicate that

$$C_{12} = 0 \tag{2.34}$$

$$C_{13} = -\frac{1}{9C_{11}^2 - 3}\left({}^{1859}/_{32}\, C_{11}^5 - {}^{445}/_4\, C_{11}^3 + {}^9/_2\, d_2 C_{11} + {}^{207}/_8\, d_1 C_{11}^2 - 3d_1\right) \tag{2.35}$$

The value of (2.34) is obtained from the solvability condition of the equation of $\Phi_4(\theta)$, and that of (2.35) from the solvability condition of the equation of $\Phi_5(\theta)$. This had necessitated a complete computation of the fourth, and a partial computation of the fifth approximations.

We shall also note that coefficient $C_{1n} = C_{1n}'/\sqrt{\pi}$ is determined from the solvability condition of the equation of $\Phi_{n+2}(\theta)$, and will appear in that condition as a linear factor in the addend $(9C_{11}^2 - 3)\, C_{1n}$; the remaining terms of this condition are known.

2.5. Approximate expression of $\Phi(\theta)$. Substituting into the first of formulas (2.1) the values of $\Phi_1(\theta)$ from (2.12), $\Phi_2(\theta)$ from (2.21), and $\Phi_3(\theta)$ from (2.30), we obtain

$$\Phi(\theta) = \varepsilon\, C_{11} \sin\theta + \varepsilon^2 C_{22} \sin 2\theta + \varepsilon^3 (C_{13} \sin\theta + C_{33} \sin 3\theta) \tag{2.36}$$

Here C_{11} is the positive real root of Eq. (2.29); C_{13} is defined by formula (2.35); C_{22} and C_{33} are expressed by formulas (2.21) and (2.31); it is also taken into account that by virtue of (2.34) and (2.31) we have $C_{12} = C_{23} = 0$.

3. Determination of the wave profile. Separating in (1.7) the real and imaginary parts and integrating, we obtain the expression of the wave profile of the parametric form as follows:

$$x = -\frac{\lambda}{2\pi}\int_0^\theta e^{-\tau(\eta)} \cos\Phi(\eta)\, d\eta, \qquad y = -\frac{\lambda}{2\pi}\int_0^\theta e^{-\tau(\eta)} \sin\Phi(\eta)\, d\eta \tag{3.1}$$

From this, using formula (2.36) for $\Phi(\eta)$, and the second of formulas (1.18) for $\tau(\eta)$, we obtain the approximate equation of the wave profile. Omitting intermediate computations, we find

$$\begin{aligned} x = -{}^1/_2\, \lambda\pi^{-1}\{\theta + \varepsilon C_{11}\sin\theta + \varepsilon^2\, {}^1/_2\, (C_{22} + C_{11}^2)\sin 2\theta + \\ + \varepsilon^3 [C_{13}\sin\theta + {}^1/_3\, (C_{33} + C_{11}C_{22} + {}^1/_6\, C_{11}^3)\sin 3\theta]\} \end{aligned} \tag{3.2}$$

$$\begin{aligned} y = {}^1/_2\, \lambda\pi^{-1}\{\varepsilon C_{11}(\cos\theta - 1) + \varepsilon^2\, {}^1/_2\, (C_{22} + {}^1/_2\, C_{11}^2)(\cos 2\theta - 1) + \\ + \varepsilon^3 [C_{13}(\cos\theta - 1) + {}^1/_3\, (C_{33} + C_{11}C_{22} + {}^1/_6\, C_{11}^3)(\cos 3\theta - 1)]\} \end{aligned} \tag{3.3}$$

In order to present the wave profile equation in the form of $y = y(x, \varepsilon)$, we eliminate θ from Eqs. (3.2) and (3.3). For this we shall attempt to express $\theta(x, \varepsilon)$ in Eq. (3.2) in the form of a function of x and ε expanded into series in powers of ε the coefficients of which are

$$\theta(x, 0) = -\frac{2\pi}{\lambda} x, \quad \left(\frac{\partial\theta}{\partial\varepsilon}\right)_{\varepsilon=0}, \ldots, \frac{1}{n!}\left(\frac{\partial^n\theta}{\partial\varepsilon^n}\right)_{\varepsilon=0}$$

the derivatives are obviously easily determined.

The equation of the profile is to be looked for in the form as follows:

$$y(x, \varepsilon) = y(x, 0) + \left(\frac{dy}{d\varepsilon}\right)_0 \varepsilon + \frac{1}{2!}\left(\frac{d^2y}{d\varepsilon^2}\right)_0 \varepsilon^2 + \frac{1}{3!}\left(\frac{d^3y}{d\varepsilon^3}\right)_0 \varepsilon^3 + \ldots \tag{3.4}$$

Here $(d^n y/d\varepsilon^n)_0 = (d^n y/d\varepsilon^n)_{\varepsilon=0}$ is assumed, and x is considered to be a parameter. Magnitudes appearing in the right side of (3.4) are determined from Eq. (3.3).

Differentiating this equation with respect to ε as a composite function, we derive the expression of $(d^n y/d\varepsilon^n)_0$ in terms of $\theta(x, 0) = -(2\pi/\lambda x)$ and $(\partial^n\theta/\partial\varepsilon^n)_{\varepsilon=0}$ with the latter magnitudes already determined as previously stated.

Having carried out all these computations, we substitute the obtained expressions of derivatives of $y(x, \varepsilon)$ into (3.4), and assuming $k = 2\pi/\lambda$, we obtain the required equation

$$y(x, \varepsilon) = k^{-1} \{\varepsilon C_{11}(\cos kx - 1) + {}^1/_2\, \varepsilon^2 C_{11}{}^2 (\cos 2kx - 1) + {}^1/_6\, \varepsilon^3 [(6C_{13} + {}^{27}/_4\, C_{11}{}^3)(\cos kx - 1) + {}^9/_4\, C_{11}{}^3 (\cos 3kx - 1)]\} \qquad (3.5)$$

By applying the procedure used for deriving Eq. (3.5), we find from (1.17) with the use of (3.2) that

$$Q(x, \varepsilon) = -\varepsilon^3 d_1 \sin kx - \varepsilon^4 (d_2 - {}^1/_2\, d_1 C_{11}) \sin 2kx + {}^1/_2\, \varepsilon^5 [({}^3/_2\, d_1 C_{11}{}^2 - 2d_2 C_{11}) \sin kx + ({}^1/_2\, d_1 C_{11}{}^2 + 2d_2 C_{11} - 2d_5) \sin 3kx] \qquad (3.6)$$

This formula confirms the statement in Sect. 1 made with respect to expression (1.17).

Note. In accordance with the conditions of this problem stated in Section 1 the coordinate origin xOy is located at the wave crest. Hence, for values of x close to zero, y must be negative. It follows from (3.5) that this is fulfilled for $C_{11} > 0$ only. On the other hand, in the case of an induced wave [2] $C_{11} = -d_1^{1/3}$, hence the condition that $C_{11} > 0$, which is valid in this case also, is fulfilled for $d_1 < 0$ only. We retain the latter inequality also for the case of a composite wave (see Sect. 1).

4. The existence and uniqueness of the solution of the problem. The following theorem may be established with the use of the Liapunov-Schmidt methods and their further developments [8].

Theorem. The system of Eqs. (1.23), (1.24) and (1.20) has for $\mu_0 = \nu_1$ a unique solution $\Phi(\theta, \varepsilon)$, $\Psi(\theta, \varepsilon)$ and $\mu(\varepsilon)$ which is small with respect to ε and continuous with respect to θ $(0 \leqslant \theta \leqslant 2\pi)$, and this solution is an analytical function of ε for $|\varepsilon| < \varepsilon_1 \leqslant \varepsilon_0$.

We shall not give here the proof of this Theorem. We shall only note that it is carried out in a manner similar to that used in [6].

This Theorem establishes the absolute and uniform convergence of expansions (2.1). The convergence of the expansion of $\tau(\theta, \varepsilon)$ follows from the Theorem in conjunction with formula (1.11). The convergence of the series expansions in powers of ε of the integrands in formulas (3.1) and (3.2) follows from the general theorems of analysis related to the substitution of expansions into expansions. The convergence of expansion (3.5) is also based on these general theorems.

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ON THE MOTION OF TWO SPHERES IN A PERFECT FLUID

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Motion of two spheres in a perfect incompressible fluid is considered. Kinetic energy and hydrodynamic forces are computed for the case when the distance between the spheres is small, in particular when the spheres touch each other. Singularities arising in the velocity field on contact of the spheres are determined.

Hicks [1] obtained the kinetic energy of the fluid for the spheres moving along the line-of-centers (the line joining the sphere centers). The kinetic energy for the case when the spheres move in the direction perpendicular to the line-of-centers and the distance separating them is much larger than their radii, is known from [2].

1. Velocity potential. Two spheres move in a perfect incompressible fluid which is at rest at infinity. The fluid motion is assumed potential. Since the problem is linear, the case when the velocities of the spheres are coplanar, is sufficient to obtain the velocity potential.

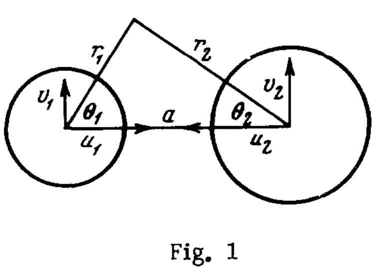

Fig. 1

We choose the spherical system of coordinates r_i, θ_i, φ_i with the origin at the center of the ith sphere ($i = 1, 2$) and the positive directions of their polar axes oriented towards the neighboring sphere (Fig. 1). Azimuthal angle φ_i is measured from the direction perpendicular to the velocities of the spheres, and the positive direction of the polar axis of the ith coordinate system is taken as positive direction of the projection u_i of the velocity on the line-of-centers. Positive directions of the projections v_1 and v_2 of the velocities of the spheres on a line perpendicular to the line-of-centers, are chosen